Chapter 1: The Reals

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Cummings' Exercises

Exercise 1.8. ⋆

(a) The following statements are equivalent

$$\begin{gathered}x \in (A \cap B)^c \\ x \notin (A \cap B) \\ x \notin A \wedge x \notin B \\ \neg(x \notin A) \vee \neg(x \notin B) \\ x \in A \vee x \in B \\ x \in A \cup B\end{gathered}$$

(b) The following statements are equivalent

$$\begin{gathered}x \in (A \cup B)^c \\ x \notin (A \cup B) \\ x \notin A \vee x \notin B \\ \neg(x \notin A) \wedge \neg(x \notin B) \\ x \in A \wedge x \in B \\ x \in A \cap B\end{gathered}$$

Exercise 1.13. ⋆

(a) We seek to prove the contrapositive. That is, if $a > b \to [\exists \varepsilon : (\varepsilon > 0) \wedge (a \geq b + \varepsilon)]$

Let $\varepsilon = a - b$. Because $a > b$, $a - b > 0$, thus $\varepsilon > 0$. Then,

$$\begin{gathered}a \geq b + \varepsilon \\ a \geq b + (a - b) \\ a \geq a\end{gathered}$$

Thus, we have found a value of ε such that $\varepsilon > 0$ and $a \geq b + \varepsilon$, completing the proof.

(b) By Note 1.11 (c), $-\varepsilon < a - b < \varepsilon$. We can split this inequality, finding that $-\varepsilon < a - b$ and $a - b < \varepsilon$.

We start by rearranging the first inequality,

$$
\begin{aligned}
a - b &> -\varepsilon \\
b - a &< \varepsilon
\end{aligned}
$$

By part (a), $b - a \leq 0$. Rearranging, $b \leq a$.

The second inequality states that $a - b < \varepsilon$ By part (a), $a - b \leq 0$. Rearranging, $a \leq b$.

Because $b \leq a$ and $a \leq b$, we conclude that $a = b$.

Exercise 1.23. ⋆

$\sup(B)$ must also be an upper bound for A. Otherwise, this implies that there exists an element in A that is larger than $\sup(B)$. Because such an element is larger than $\sup(B)$, it cannot be in B, thus it cannot be in A because $A \subseteq B$.

$\sup(A) \leq$ all upper bounds of A. To prove this, assume that there exists a smaller upper bound of A, α such that $\alpha < \sup(A)$. Then, let $\varepsilon = \sup(A) - \alpha$. Because $\alpha < \sup(A)$, $\varepsilon > 0$. Then, there exists an ε such that $\sup(A) - \varepsilon = \alpha$, contradicting the definition of $\sup(A)$.

Thus, $\sup(A) \leq \sup(B)$.

Exercise 1.30.

(a) By the definition of $\sup(B)$, $\forall \varepsilon : \varepsilon > 0 \to \sup(B) - \varepsilon$ is not an upper bound of B.

Then, $\forall \varepsilon : \varepsilon > 0 \to [\exists b : (b \in B) \wedge (b > \sup(B) - \varepsilon)]$.

Let $\varepsilon = \sup(B) - \sup(A)$. Because $\sup(A) < \sup(B)$, $\varepsilon > 0$. Thus, $\exists b : (b \in B) \wedge (b > \sup(A))$.

(b) Let $A = B = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$. Then, $\sup(A) = \sup(B) = 1 \notin B$

Exercise 1.34. ⋆

Let $P(n)$ be the statement that $\cap_{i=1}^{n} I_i = I_n \neq \emptyset$.

This is true for $P(1)$, as I_1 is not empty.

By definition of an interval, I_{n+1} is not empty. Additionally, $I_{n+1} \subseteq I_n$ so $I_{n+1} \cap I_n = I_{n+1}$. Then, the following statements are equivalent:

$$
\begin{gathered}
\cap_{i=1}^{n+1} I_i \\
\cap_{i=1}^{n} I_i \cap I_{n+1} \\
\underbrace{I_n}_{\text{IH}} \cap I_{n+1} \\
I_{n+1}
\end{gathered}
$$

MIT Exercises

Exercise 3 from Homework 1

Let $P(n)$ be the statement that $\forall A : (|A| = n) \to (|\mathscr{P}(A)| = 2^n)$

For $P(0)$, $A = \emptyset$. Then, $\mathscr{P}(\emptyset) = 2^0 = 1$.

Let $|A| = n + 1$. Then, choose any element $a \in A$. Then, let $B = A \setminus \{a\}$, such that $A = B \cup \{a\}$. Because $|B| = n$, by the inductive hypothesis, $|\mathscr{P}(B)| = 2^n$.

By definition of $\mathscr{P}$, $\mathscr{P}(A) = \mathscr{P}(B) \cup \{b \cup \{a\} : b \in \mathscr{P}(B)\}$. Thus,

$$\begin{aligned} |\mathscr{P}(A)| &= |\mathscr{P}(B)| + |\mathscr{P}(B)| \\ &= 2^n + 2^n \\ &= 2^{n+1} \end{aligned}$$

Exercise 6 from Homework 1

(a) To compute the first expression,

$$\begin{aligned} f\left(\frac{4}{15}\right) &= f\left(\frac{2^2}{3 \cdot 5}\right) \\ &= 2^{2 \cdot 2} \cdot 3^{2 \cdot 1 - 1} \cdot 5^{2 \cdot 1 - 1} \\ &\boxed{= 240} \end{aligned}$$

To compute the value of q,

$$\begin{aligned} 108 &= 2^2 \cdot 3^3 \\ &= 2^{2 \cdot 1} \cdot 3^{2 \cdot 2 - 1} \end{aligned}$$

Thus, $q = \frac{2^1}{3^2} = \frac{2}{9}$

(b) By †, all $f(q) \in \mathbb{N}$ have a unique prime factorization in the form $f(q) = p_1^{n_1} p_2^{n_2} ... p_N^{n_N}$. Each n_i can be written in the form of $n_i = 2r_i$ if even or $n_i = 2s_i - 1$ if odd. By the **Theorem**, there must exist a unique q with these unique exponents, proving that f is a bijection.

Exercise 1 from Homework 2

We are given that $z - y \in P$ and $x \notin P$ (and thus $-x \in P$). Then,

$$\begin{aligned} z - y &\in P \\ -x(z - y) &\in P \\ -(xz - xy) &\in P \end{aligned}$$

Then, $-(xz - xy) \in P$, thus $xz - xy \notin P$. By definition, $xz - xy < 0$.

Exercise 2 from Homework 2

Let $P(A)$ be the statement that if A is a nonempty finite subset of ordered field S, then A is bounded and that $\inf(A), \sup(A) \in A$.

If A is the singleton set, then there exists a single member $a \in A$. Then, $\sup(A) = \inf(A) = a$. $a \leq a \geq a$, so it bounds the single member in the set. To show that it is the least upper bound of A, for $\varepsilon > 0$, $a - \varepsilon < a$, thus, there is no lesser upper bound. Similarly, $a + \varepsilon > a$, so there is no greater lower bound.

For all other nonempty sets A, it can be partitioned into two disjoint subsets such that $A = B \cup \{x\}$, where $B = A \setminus \{x\}$. By the inductive hypothesis, $\sup(B)$ and $\inf(B)$ exist and are in B.

Case 1: $\inf(B) \leq x$
$\inf(B)$ and $\sup(B)$ bound all elements of B. $\inf(B) \leq x$ as well, thus it is a lower bound for all elements of A. There exists some $b \in B = \inf(B) \leq x$. For all $\varepsilon > 0$, $\inf(B) + \varepsilon = b + \varepsilon < .\neg b$, so that $\inf(B)$ is also the greatest lower bound for A. Then, $\inf(B) = \inf(A) \in A$.

Case 2: $\inf(B) > x$
Let $\inf(B) = b \in B$. Then, x is smaller than the smallest member of B, b, and it is the smallest member in A. Therefore, x is a lower bound for A and B.

A symmetrical line of reasoning applies to x and $\sup(B)$.